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BIMEROMORPHIC GEOMETRY OF NORMAL GORENSTEIN SURFACES

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The purpose of this note is to provide a "bimeromorphic geometry" in the category of normal Gorenstein surfaces. Details will be discussed elsewhere.

§1. Normal Gorenstein Surfaces

An isolated singularity y of a surface Y is said to be Gorenstein if there exists a neighborhood U of y satisfying (i) the restriction map $O(U) \longrightarrow O(U \setminus y)$ is bijective (i.e., y is normal), (ii) $\exists$ a nowhere vanishing holomorphic 2-form on $U \setminus y$.[1)]

Example. Hypersurface singularities are always Gorenstein. In particular, the following singularities are called rational double points.

1) In the 2-dimensional case, we have the equivalence:

$$\left\{\begin{array}{c}\text{an isolated Gorenstein}\\ \text{singularity}\end{array}\right\} = \left\{\begin{array}{c}\text{a normal Gorenstein}\\ \text{singularity}\end{array}\right\}$$

For ring theoretic aspects, we refer to the expository article by S.Goto:On Gorenstein rings(in Japanese), Sugaku 31(1979).

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A_n : $x^2+y^2+z^{n+1}=0$ $n \geqq 1$

D_n : $x^2+y^2z+z^{n-1}=0$ $n \geqq 4$

E_6 : $x^2+y^3+z^4=0$

E_7 : $x^2+y^3+yz^3=0$

E_8 : $x^2+y^3+z^5=0$

In what follows, by a normal Gorenstein surface we shall mean a 2-dimensional, reduced, irreducible compact complex space with isolated Gorenstein singularities.

Let Y be a normal Gorenstein surface. The singular locus $\mathrm{Sing}(Y)$ consists of finite points $\{y_1,\ldots,y_s\}$. The sheaf of holomorphic 2-forms Ω^2 on $Y\setminus\mathrm{Sing}(Y)$ extends naturally as an invertible sheaf ω_Y to Y. This sheaf is nothing but the Grothendieck dualizing sheaf of Y. We denote by K_Y a canonical divisor (bundle, in general) of Y corresponding to the sheaf ω_Y. For positive integers m, we define the arithmetic m-genus by

$$\overline{P}_m(Y)=\dim H^0(Y,\ \omega_Y^{\otimes m}).\ ^{2)}$$

2) We borrowed the bar notation from Wilson, P.M.H.: The arithmetic plurigenera of surfaces. Math. Proc. Camb. Phil. Soc. 85 (1979), 25-31. If there is no danger of confusion, we may omit the bar as in a paper of Brenton, L.: On singular complex surfaces with vanishing geometric genus and pararational singularities. Compositio Math. 43 (1981), 297-315.

§2. Local Properties (For instance, see [8])

Let $\pi: X \longrightarrow Y$ be the minimal resolution in the sense that there exist no exceptional curves of the first kind in $\pi^{-1}(\mathrm{Sing}(Y))$. Then there exists a unique effective divisor Δ such that

$$\pi^*\omega_Y \cong O(K_X+\Delta).$$

Furthermore if $\Delta>0$, we easily find the property:

$$\omega_\Delta \cong O_\Delta.\ ^{3)}$$

Let $\Delta=\Sigma\Delta_i$ be the decomposition so that each Δ_i is supported in $\pi^{-1}(y_i)$. We have $\Delta_i=0$ if and only if y_i is a rational double point. In case $\Delta_i>0$, we have $\mathrm{Supp}(\Delta_i)=\pi^{-1}(y_i)$ and we find

$$\dim(R^1\pi_*O_X)_{y_i} = \dim H^1(\Delta_i, O_{\Delta_i}).$$

So y_i is a rational singularity if and only if it is a rational double point.

Remark. The converse process is possible. Let Δ be a connected curve on a non-singular surface X satisfying

(i) the intersection matrix of irreducible components of Δ is negative definite,

(ii) $\omega_\Delta \cong O_\Delta$.

Then Δ can be contracted to an isolated Gorenstein singularity (Laufer[6], Bǎdescu[4]).

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3) By definition $\omega_\Delta = \mathrm{Ext}^1_{O_X}(O_\Delta, \omega_X)$ which is isomorphic to the sheaf $O(K_X+\Delta)\otimes O_\Delta$.

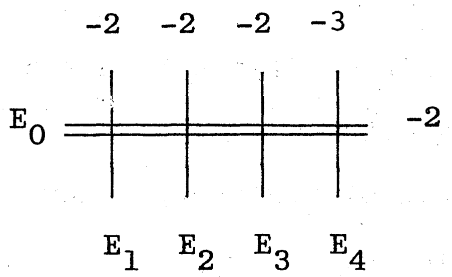
The divisors Δ_i can be calculated as follows. Let $E_1,\ldots,E_r$ be the irreducible components of $\pi^{-1}(y_i)$. If $\Delta_i=\Sigma n_j E_j$, then $(n_1,\ldots,n_r)$ is a solution of the equations:

$$(K_X+\Sigma n_j E_j)E_k=0 \quad \text{for } k=1,\ldots,r.$$

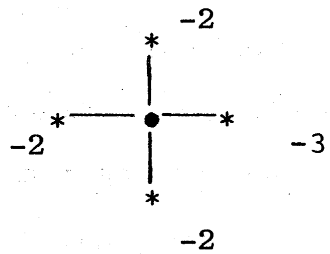
Since the intersection matrix $(E_j E_k)$ is negative definite, the solution is unique. We also get the non-negativeness of each n_j from the fact that the resolution is minimal. Furthermore if y_i is Gorenstein, the solution must be integral.

Example. For a minimally elliptic singularity(i.e., the case where $\dim H^1(\Delta_i,\mathcal{O}_{\Delta_i})=1$), the divisor Δ_i coincides with the fundamental cycle of y_i(cf.[6]).

Equation	Dual Graph
$x^2=y^3+z^9$	$A_{1,-2,-2,-2,-3}$



all components are $\mathbb{P}_1$



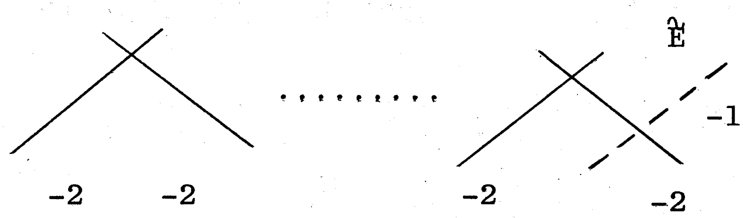
In this example, we have

$$\Delta=2E_0+E_1+E_2+E_3+E_4.$$

§3. Generalized Blowing Ups

We shall introduce (generalized) blowing ups for normal Gorenstein surfaces. Namely, we define a blowing up $\tilde{Y}$ of Y at a point y in the following way.

Case(i). y is a non-singular point. Take a sequence of usual blowing ups n-times so that the exceptional curves form

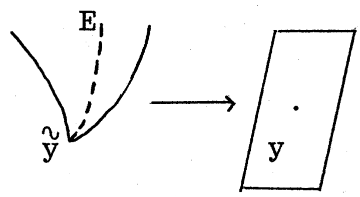


By contracting the (-2) curves [4] to a rational double point $\tilde{y}$ (type A_{n-1}), we obtain a blowing up $\mu:\tilde{Y}\longrightarrow Y$. In case n=1, this is nothing but the usual one. If $\tilde{\tilde{E}}$ denotes the (-1) curve, then μ contracts the curve $\tilde{E}=\tilde{\pi}(\tilde{\tilde{E}})$ to y. We find

$$\omega_{\tilde{Y}}\cong\mu^*\omega_Y\otimes O(E).$$

In this case E is a Cartier divisor.

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{\mu}"] \arrow[d, "\tilde{\pi}"'] & X \arrow[d, equal, "\text{near } y"] \\ \tilde{Y} \arrow[r, "\mu"'] & Y \end{tikzcd}$$



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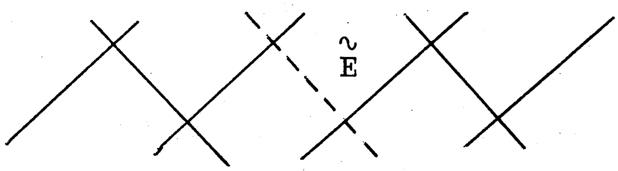
4) For simplicity's sake, we use the terminology:

$$(-k)\text{ curve}=\left\{\begin{matrix}\text{a non-singular rational curve}\\ \text{with self-intersection number } -k\end{matrix}\right\}$$

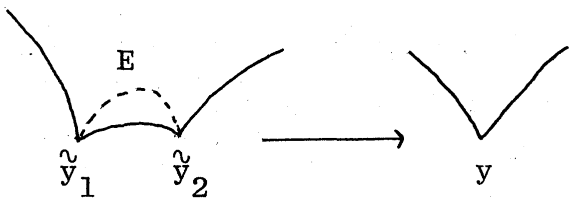
<u>Case</u>(ii). y is a rational double point. Then $\pi^{-1}(y)$ is a union of (-2) curves. Let $\tilde{\pi}:X\longrightarrow\tilde{Y}$ be the contraction of these curves except one curve $\tilde{E}$. We thus obtain a blowing up $\mu:\tilde{Y}\longrightarrow Y$ which contracts the curve $E=\tilde{\pi}(\tilde{E})$ to y. In this case E need not be a Cartier divisor. We have the formula:

$$\omega_{\tilde{Y}}\cong\mu^*\omega_Y.$$

<u>Example</u>. Suppose $\pi^{-1}(y)$ is of type A_7. Let $\tilde{E}$ be the central curve.



In Y, we have two points $\tilde{y}_1$, $\tilde{y}_2$ of type A_3.



<u>Case</u>(iii). y is a non-rational singularity. Choose a point x_1 in Δ. Let $\tilde{\mu}_1:X_1\longrightarrow X$ be the usual blowing up at x_1. Put $\Delta_1=\tilde{\mu}_1^*\Delta-E_1$ where the E_1 is the exceptional curve. If the multiplicity of Δ at x_1 is greater than or equal to two, we have $E_1\subset\Delta_1$. Choose again a point x_2 in E_1 and let $\tilde{\mu}_2:X_2\longrightarrow X_1$ be the usual blowing up at x_2. We continue in this way so that $\nu_1\geqq 2,\ldots,$ $\nu_{n-1}\geqq 2$, but $\nu_n=1$ where each ν_i denotes the multiplicity of Δ_{i-1} at x_i. Then $\tilde{\Delta}=\tilde{\mu}_n^*\Delta_{n-1}-E_n$ does not contain $\tilde{E}=E_n$, from which we infer that $\tilde{\Delta}$ contains no (-1) curves. By construction $\omega_{\tilde{\Delta}}\cong O_{\tilde{\Delta}}$. Therefore $\tilde{\Delta}$ can be contracted to a Gorenstein singularity $\tilde{y}$. Thus we obtain a blowing up $\mu:\tilde{Y}\longrightarrow Y$ where the curve $E=\tilde{\pi}(\tilde{E})$ is

contracted to y. We find

$$\omega_{\tilde{Y}} \cong \mu^*\omega_Y.$$

Note that E need not be a Cartier divisor.

$$\begin{tikzcd}
\tilde{X}=X_n \arrow[r] \arrow[d, "\tilde{\pi}"'] & \cdots\cdots\cdots\cdots \arrow[r] & X_1 \arrow[r] & X \arrow[d, "\pi"] \\
\tilde{Y} \arrow[rrr, "\mu"'] & & & Y
\end{tikzcd}$$

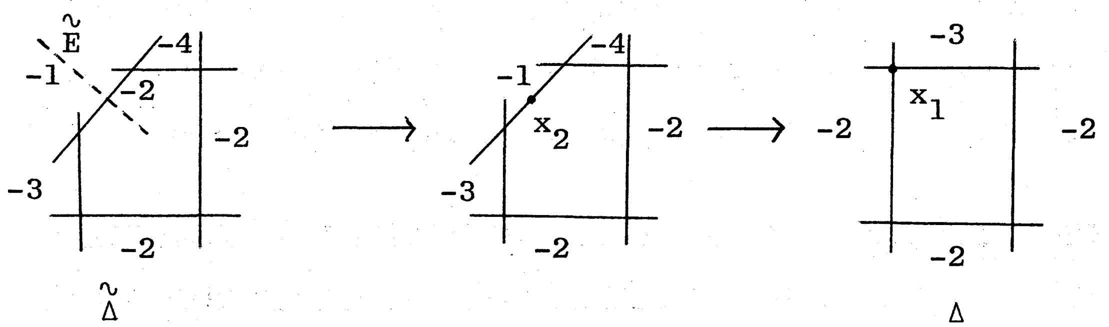
Remark. The above defined blowing ups are characterized by the following properties (cf. §4):

(i) $\tilde{Y}$ is again a normal Gorenstein surface,

(ii) μ contracts an irreducible curve E to a point y such that $\mu: \tilde{Y}\backslash E \longrightarrow Y\backslash y$ is an isomorphism,

(iii) $\mu_*\omega_{\tilde{Y}} \cong \omega_Y$.

Example.



$z(xy+z^2)=x^5+y^6$ $\qquad\qquad$ $z^2=(y+x^2)(y^2+x^8)$

We borrowed the equations from [6]. All curves are non-singular rational curves. In this example, $\nu_1=2$, $\nu_2=1$.

§4. Factorization Theorem

Let Y, Y' be two normal Gorenstein surfaces. A bimeromorphic morphism $f:Y\longrightarrow Y'$ is said to be canonical if $f_*\omega_Y\cong\omega_{Y'}$.

Lemma. *Let $f:Y\longrightarrow Y'$ be a canonical bimeromorphic morphism between normal Gorenstein surfaces. Then*

$$f_*\omega_Y^{\otimes m}\cong\omega_{Y'}^{\otimes m} \qquad \textit{for } m>0.$$

Proof. We infer from the hypothesis that $\omega_Y\cong f^*\omega_{Y'}\otimes O(G)$ where the effective Cartier divisor G is contracted to points by f. It follows that $\omega_Y^{\otimes m}\cong f^*\omega_{Y'}^{\otimes m}\otimes O(mG)$, which deduces the required result. Q.E.D.

Corollary. *We have*

$$\overline{P}_m(Y)=\overline{P}_m(Y').$$

In order to develop a geometry with invariants $\overline{P}_m$, it is therefore natural to restrict ouselves to canonical bimeromorphic morphisms. Our main observation is the following

Theorem. *Any canonical bimeromorphic morphism between normal Gorenstein surfaces can be factored into a sequence of blowing ups* (as defined in §3).

Remark. Between non-singular surfaces, a bimeromorphic morphism is automatically canonical. So the above theorem generalizes the well known fact that any bimeromorphic morphism between non-singular surfaces can be factored into a sequence of blowing ups.

As a consequence, we propose the following

Definition. Two normal Gorenstein surfaces Y and Y' are in the same "bimeromorphic class" if they are connected by blowing ups:

$$\mu_r\circ\cdots\circ\mu_1(Y)=\mu'_s\circ\cdots\circ\mu'_1(Y'),$$

where μ_i, μ'_j are blowing ups defined in §3.

Remark. The following conditions are equivalent for bimeromorphic morphisms $f:Y\longrightarrow Y'$ between normal Gorenstein surfaces.

(i) f is canonical,

(ii) $R^1f_*O_Y=0$.

This follows from the Grothendieck duality(Kempf's argument in [5], p.49) via the Grauert-Riemenschneider vanishing theorem for f (See §5).

§5. Generalities

Finally we collect some general results concerning normal Gornstein surfaces. Let L be a line bundle on a normal Gorenstein surface Y.

I Riemann-Roch Theorem([3])

$$\chi(O(L))=\frac{1}{2}(L^2-KL)+\chi(O_Y),$$

II <u>Serre Duality</u>

$$H^i(Y,O(K+L))\cong H^{2-i}(Y,O(-L))^{\vee},$$

III <u>Kodaira-Ramanujam Vanishing Theorem</u>

If (i) $L^2>0$, (ii) $LC\geqq 0$ for all curves C on Y, then [5)]

$$H^i(Y,O(K+L))=0 \text{ for } i>0,$$

IV <u>Grauert-Riemenschneider Vanishing Theorem</u> (cf.[4])

If $f:Y\longrightarrow Y'$ is a surjective morphism between normal Gorenstein surfaces, then

$$R^i f_*\omega_Y=0 \text{ for } i>0, \quad {}^{6)}$$

V <u>Classification Theory</u>

In the Moišezon case (i.e., Y has two algebraically independent meromorphic functions, cf.[1]), we have an Enriques type classification. We refer the reader to [8].

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5) If such a bundle L exists, then Y is a Moišezon surface.

6) In general, Vanishing Theorems III, IV are valid for normal surfaces. We can employ the fact that the injective trace map

$$\pi_*\omega_X \hookrightarrow \omega_Y$$

has a cokernel supported on isolated singularities. Here X denotes the minimal resolution of Y. Hence, for instance

$$H^i(Y,\pi_*\omega_X\otimes O(L))\longrightarrow H^i(Y,\omega_Y\otimes O(L))\longrightarrow 0$$

for i>0.

References

[1] Artin,M.:Algebraic Spaces. Yale Math. Monographs 3, Yale Univ. Press 1971

[2] Bădescu,L.:Applications of the Grothendieck duality theory to the study of normal isolated singularities. Rev. Roum. Pures et Appl. 24 (1979), 673-689

[3] Brenton,L.:On the Riemann-Roch equation for singular complex surfaces. Pacific J. Math. 71 (1977), 299-312

[4] Grauert,H. and Riemenschneider,O.:Verschwindungssätze für analytische Kohomologiegruppe auf komplexen Räumen. Invent. Math. 11 (1970), 263-292

[5] Kempf,G. et al.:Toroidal Embeddings I. Lecture Notes in Mathematics Vol. 339, Springer 1973

[6] Laufer,H.B.:On minimally elliptic singularities. Amer. J. Math. 99 (1975), 1275-1295

[7] Reid,M.:Canonical 3-folds. Proc. of "Journées de Géométrie Algebrique" Angers 1979 pp.273-310, Sijthoff and Nordhoff 1980

[8] Sakai,F.:Enriques classification of normal Gorenstein surfaces. To appear in Amer. J. Math.

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